The Fuchsian differential equation of the square lattice Ising model $\chi^{(3)}$ susceptibility

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Abstract. Using an expansion method in the variables x_i that appear in the $(n-1)$-dimensional integrals representing the n-particle contribution to the Ising square lattice model susceptibility χ, we generate a long series of coefficients for the 3-particle contribution $\chi^{(3)}$, using a N^4 polynomial time algorithm. We give the Fuchsian differential equation of order seven for $\chi^{(3)}$ that reproduces all the terms of our long series. An analysis of the properties of this Fuchsian differential equation is performed.

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1. Introduction

Since the work of T.T. Wu *et al.* [1], it is known that the expansion in n-particle contributions to the zero field susceptibility of the square lattice Ising model at temperature T can be written as a sum:

$$\chi(T) = \sum_{n=1}^{\infty} \chi^{(n)}(T) \qquad (1)$$

of $(n-1)$-dimensional integrals [2, 3, 4, 5, 6, 7], the sum being restricted to odd (respectively even) n for the high (respectively low) temperature case. While the first contribution in the sum, $\chi^{(1)}$, is obtained directly without integration, and the second one, $\chi^{(2)}$, is given in terms of elliptic integrals, *no closed forms* for the higher order contributions are known despite the well-defined forms of these $(n-1)$-dimensional integrals. The $\chi^{(n)}$'s are $(n-1)$-dimensional integrals of holonomic (algebraic) expressions, and are consequently *holonomic, or "D-finite"*: they are solutions of *finite*

order differential equations with polynomial coefficients [8, 9]. Unfortunately, such theorems of holonomy theory *do not give any lower or upper bound, or hint*, as to the order of the differential equation satisfied by a given $\chi^{(n)}$.

As far as singular points are concerned (physical or non-physical singularities in the complex plane), and besides the known $s = \pm 1$ and $s = \pm i$ singularities, B. Nickel showed [6] that $\chi^{(2n+1)}$ is singular‡ for the following finite values of $s = sh(2J/kT)$ lying on the *unit circle* ($m = k = 0$ excluded):

$$\begin{aligned} 2 \cdot \left(s + \frac{1}{s}\right) &= u^k + \frac{1}{u^k} + u^m + \frac{1}{u^m} \\ u^{2n+1} &= 1, \qquad\qquad -n \le m, k \le n \end{aligned} \qquad (2)$$

When n increases, the singularities of the higher-particle components of $\chi(s)$ *accumulate on the unit circle* $|s| = 1$. The existence of such a natural boundary for the total $\chi(s)$, shows that $\chi(s)$ is *not D-finite* (holonomic) as a function of s. To understand the analytical structure of such a *transcendental* function, it is thus crucial to better understand the analytical structure of the "holonomic" $\chi^{(n)}$'s, and, as a first step, to find the still unknown differential equation verified by the 3-particle contribution $\chi^{(3)}$.

A significant amount of work had already been performed to generate isotropic series coefficients for $\chi^{(n)}$ (by B. Nickel [6, 7] up to order 116, then to order 257 by A.J. Guttmann§ *et al.*). More recently, W. Orrick *et al.* [10], have generated coefficients‖ of $\chi(s)$ up to order 323 and 646 for high and low temperature series in s, using Perk's *non-linear Painlevé difference equations* for the correlation functions [10, 11, 12, 13, 14]. As a consequence of this non-linear Painlevé difference equation and the associated remarkable recursion on the coefficients, the computer algorithm had a $O(N^6)$ polynomial growth of the calculation of the series expansion instead of an exponential growth that one would expect at first sight. However, in such a non-linear, non-holonomic, Painlevé-oriented approach, one obtains results directly for the total susceptibility $\chi(s)$ which does not satisfy any linear differential equation, and thus prevents the easily disentangling of the contributions of the various holonomic $\chi^{(n)}$'s.

In contrast, we develop here, a *strictly holonomic approach*. This approach enabled us to get 490 coefficients of the series expansion of $\chi^{(3)}$, from which we have deduced the *Fuchsian differential equation* of order *seven* satisfied by $\chi^{(3)}$. This Fuchsian differential equation presents a large set of remarkable properties and structures that are briefly sketched here and will be analyzed in more details in forthcoming publications, along with the analytical behavior of the solutions. The method used in this paper to obtain the Fuchsian differential equation is not specific of the third contribution $\chi^{(3)}$ and can be generalized, mutatis mutandis, to the other $\chi^{(n)}$, $(n > 3)$ without any drastic changes in the mathematical framework¶.

‡ The singularities being logarithmic branch points of order $\epsilon^{2n(n+1)-1} \cdot ln(\epsilon)$ with $\epsilon = 1 - s/s_i$ where s_i is one of the solutions of (2).
§ Private communication.
‖ The short-distance terms were shown to have the form $(T - T_c)^p \cdot (log|T - T_c|)^q$ with $p \ge q^2$.
¶ But certainly requiring larger computer calculations.

2. Generating the series for $\chi^{(3)}$

Similarly to Nickel's papers [6, 7], we start using the multiple integral form of the $\chi^{(n)}$'s. We focus on $\chi^{(3)}$ and consider the double integral:

$$\chi^{(3)}(s) = \frac{(1-s^4)^{1/4}}{s} \;\; \tilde{\chi}^{(3)}(s)$$
$$\tilde{\chi}^{(3)}(s) = \frac{1}{4\pi^2} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \;\; \tilde{y}_1 \; \tilde{y}_2 \; \tilde{y}_3 \left(\frac{1+\tilde{x}_1\tilde{x}_2\tilde{x}_3}{1-\tilde{x}_1\tilde{x}_2\tilde{x}_3} \right) H^{(3)} \qquad (3)$$

with:

$$\tilde{x}_j = \frac{s}{1+s^2 - s\cos\phi_j + \sqrt{(1+s^2-s\cos\phi_j)^2 - s^2}}, \qquad (4)$$
$$\tilde{y}_j = \frac{s}{\sqrt{(1+s^2-s\cos\phi_j)^2-s^2}}, \qquad j=1,2,3, \quad \phi_1+\phi_2+\phi_3 = 0$$

Many forms [6, 7] for $H^{(3)}$ may be taken and are equivalent for integration purposes, e.g.,

$$H^{(3)} = f_{23} \left(f_{31} + \frac{f_{23}}{2} \right), \qquad f_{ij} = (\sin\phi_i - \sin\phi_j) \; \frac{\tilde{x}_i\tilde{x}_j}{1-\tilde{x}_i\tilde{x}_j} \qquad (5)$$

It is straightforward to see that $\tilde{\chi}^{(3)}$ is only a function of the variable $w = \frac{1}{2}s/(1+s^2)$. From now on, we thus focus on $\tilde{\chi}^{(3)}$ seen as a function of the well-suited variable w instead of s. We may expand the integrand in (3) in this variable w and integrate the angular part. For $\tilde{\chi}^{(3)}$, this would mean there are 18 sums to carry out. We, instead, expand the integrand in the variable $\tilde{x}$ of (4), where we succeeded in deriving remarkable formulas for $\tilde{y}\,\tilde{x}^n$ carrying one summation index. As a consequence, we are able to write $\tilde{\chi}^{(3)}(w)$ as a fully integrated expansion. Our algorithm runs in a *polynomial* time calculations (namely $O(N^4)$). In contrast with Orrick *et al.* calculations [10], this calculation is not based on any recursion: it allows one to obtain any given coefficient *separately without requiring the storage of all the previous data*. The details of these calculations, tricks$^+$, and of this program will be given elsewhere [15].

At present, we have obtained* the expansion of $\tilde{\chi}^{(3)}$ up to w^{490}. As expected, this expansion is in agreement with the previous results published by Nickel [6, 7] using a numerical method of integration, as well as the improved unpublished results (up to w^{257}) by A.J. Guttmann ♯ *et al.* In terms of the well-suited variable w, the first terms of the expansion of $\tilde{\chi}^{(3)}(w)$ read :

$$\frac{\tilde{\chi}^{(3)}(w)}{8} \;=\; w^9 + 36\,w^{11} + 4\,w^{12} + 884\,w^{13} + 196\,w^{14}$$
$$+ 18532\,w^{15} + 6084\,w^{16} \,+ \cdots \qquad (6)$$

3. The Fuchsian differential equation satisfied by $\tilde{\chi}^{(3)}(w)$

Given the expansion of $\tilde{\chi}^{(3)}(w)$ up to w^{490}, the next step will be to encode all the numbers in this long series into a linear differential equation. Note that such an equation should exist [8, 9] though, its order is unknown. Using a dedicated program

$^+$ For instance, our calculations also underline the important role played by hypergeometric functions.
* We can get a new coefficient every two days.
♯ Private communication.

for searching for such a linear differential equation with polynomial coefficients in w and steadily increasing the order, we succeeded finally in finding the following linear differential equation of order *seven* satisfied by the 490 terms we have calculated for $\tilde{\chi}^{(3)}$:

$$\sum_{n=0}^{7} a_n \cdot \frac{d^n}{dw^n} F(w) \;=\; 0 \tag{7}$$

with :

$$\begin{aligned}
a_7 \;=\;& \\
& w^7 \cdot (1-w)\,(1+2\,w)\,(1-4\,w)^5\,(1+4\,w)^3\,\left(1+3\,w+4\,w^2\right)\,P_7(w) \\
a_6 \;=\;& w^6 \cdot (1-4\,w)^4\,(1+4\,w)^2\,P_6(w) \\
a_5 \;=\;& w^5 \cdot (1-4\,w)^3\,(1+4\,w)\,P_5(w) \\
a_4 \;=\;& w^4 \cdot (1-4\,w)^2\,P_4(w), \qquad\qquad a_3 \;=\; w^3 \cdot (1-4\,w)\,P_3(w) \\
a_2 \;=\;& w^2\,P_2(w), \qquad\quad a_1 \;=\; w\,P_1(w), \qquad\quad a_0 \;=\; P_0(w)
\end{aligned} \tag{8}$$

where $P_7(w), P_6(w) \cdots, P_0(w)$ are polynomials of degree respectively 28, 34, 36, 38, 39, 40, 40 and 36 in w [15]. These polynomials are too large to be given here.

Note that the series of $\tilde{\chi}^{(3)}(w)$ up to order 359, is sufficient to identify the differential equation when the order $q = 7$ and the successive degrees $47, 46, 45, 44, 43, 42, 41$ and 36 of the polynomials in front of the $F^{(n)}(w)$'s are imposed. Inside this framework, and since our series has 490 coefficients, we have here 131 verifications of the correctness of this differential equation.

At this point, let us remark that if one had worked with variable s, instead of w, the series in s up to order 699 would be needed in order to obtain the seventh order differential equation satisfied by $\tilde{\chi}^{(3)}(s)$.

Since the singular points of this differential equation correspond to the roots of the polynomial corresponding to the highest order derivative $F^{(7)}(w)$, namely a_7, we give the exact expression of $P_7(w)$:

$$\begin{aligned}
P_7(w) \;=\;& 1568 + 15638\,w - 565286\,w^2 - 276893\,w^3 \\
& + 34839063\,w^4 + 100696470\,w^5 - 1203580072\,w^6 \\
& - 5514282112\,w^7 + 18005067728\,w^8 + 110343422816\,w^9 \\
& - 140604884224\,w^{10} - 1825536178688\,w^{11} + 920432273408\,w^{12} \\
& + 28913052344320\,w^{13} + 38181758402560\,w^{14} \\
& - 112307319603200\,w^{15} - 544140071665664\,w^{16} \\
& - 1144172108054528\,w^{17} - 1027222993371136\,w^{18} \\
& - 1992177026596864\,w^{19} - 2948885085421568\,w^{20} \\
& + 2211524294737920\,w^{21} + 8204389336481792\,w^{22} \\
& + 675795924156416\,w^{23} - 2882636020187136\,w^{24} \\
& - 5364860829302784\,w^{25} - 222238787764224\,w^{26} \\
& + 158329674399744\,w^{27} + 39582418599936\,w^{28}
\end{aligned} \tag{9}$$

The differential equation (7) is an equation of the *Fuchsian type since there are no singular points, finite or infinite, other than regular singular points.* With this property, using Frobenius method [16], it is straightforward to obtain, from the indicial

equation, the critical exponents, in w, for each regular singular point. These are given in Table 1.

$w=0$	$s=0$	$\rho=9,3,2,2,1,1,1$
$w=-1/4$	$s=-1$	$\rho=3,2,1,0,0,0,-1/2$
$w=1/4$	$s=1$	$\rho=1,0,0,0,-1,-1,-3/2$
$w=-1/2$	$1+s+s^2=0$	$\rho=5,4,3,3,2,1,0$
$w=1$	$2-s+2s^2=0$	$\rho=5,4,3,3,2,1,0$
$1+3\,w+4\,w^2=0$	$(2s^2+s+1)(s^2+s+2)=0$	$\rho=5,4,3,2,1,1,0$
$1/w=0$	$1+s^2=0$	$\rho=3,2,1,1,1,0,0$
$w=w_P$, 28 roots	$s=s_P$, 56 roots	$\rho=7,5,4,3,2,1,0$

Table 1: Critical exponents for each regular singular point. w_P is any of the 28 roots of $P_7(w)$. We have also shown the corresponding roots in the s variable††.

At this point, it is worth recalling the Fuchsian relation on Fuchsian type equations. Denoting $w_1,\ w_2, \cdots,\ w_m,\ w_{m+1}\ =\ \infty$, the regular singular points of a Fuchsian type equation of order q and $\rho_{j,1},\ \cdots, \rho_{j,q}$ $(j\ =\ 1, \cdots, m+1)$ the q roots of the indicial equation corresponding to each regular singular point w_j, the following Fuchsian relation holds :

$$\sum_{j=1}^{m+1} \sum_{k=1}^{q} \rho_{j,k} \ = \ \frac{(m-1)\ q\ (q-1)}{2} \qquad (10)$$

The number of regular singular points is here $m+1\ =\ 36$ corresponding respectively to the 28 roots of P_7, the two roots of $1+3w\ +4\,w^2$, five regular singular points, and the point at infinity $w\ =\ \infty$. The Fuchsian relation is actually verified here with $q\ =7,\ m\ =\ 35$, the sum of all the $\rho_{j,k}$'s being actually 714.

Let us comment on the singularities appearing in (7). The well-known ferromagnetic and antiferromagnetic critical points correspond respectively to $w = 1/4$ and $w\ =\ -1/4$, while $w\ =\ 0$ corresponds to the zero or infinite temperature points. The value $w\ =\ \infty$, or $s\ =\ \pm i$, is known as a non-physical singularity. The values $w\ =\ 1$ and $w\ =\ -1/2$ correspond to the non-physical singularities (11) described by Nickel for $n=1$ $(m=k=0$ excluded) :

$$\frac{1}{w} \ = \ u^k \ + \ \frac{1}{u^k} \ + \ u^m \ + \ \frac{1}{u^m}$$
$$u^3 \ = \ 1, \qquad\qquad -1 \ \le \ m,\ k \ \le \ 1 \qquad (11)$$

Furthermore, besides the known singularities mentioned above, we remark the occurrence of the roots of the polynomial P_7 of degree 28 in w, and the two *quadratic*

††In the variable s the local exponents for $w=\pm 1/4$ are twice those given.

numbers $1 + 3\,w + 4\,w^2 \; = \; 0$ which are not of the form (11). The two quadratic numbers are not on the s-unit circle : $|s| = \sqrt{2}$ and $|s| = 1/\sqrt{2}$.

To analyze the local solutions of the differential equation, let us recall that, in general, it is known [16] that for a set of k local exponents such that the difference, in absolute value, between any two of them is an integer, the local solutions may contain logarithmic terms up to $\log^{k-1}$. In fact, the Fuchsian equation (7) is such that the occurrence of logarithmic terms, in the local solutions near a given regular singular point, is due only to the occurrence of multiple roots in the corresponding indicial equation: a root of multiplicity p inducing logarithmic terms up to $\log^{p-1}$, i.e., *at most* $\log^2$ for (7), as it is shown also in the monodromy matrices (see below).

More precisely for P_7, near any of its roots, all the local solutions carry *no logarithmic terms* and *are analytical* since the exponents *are all positive integers*. The roots of P_7 are thus *apparent singularities* [16] of the Fuchsian equation (7). Note that the "apparent" character of P_7, means that, to ensure the absence of logarithmic terms in the general solution, there are $q \cdot (q-1)/2 = 21$ relations between the various P_n's at each root of P_7 (for more details see [16]).

The two unexpected quadratic numbers correspond to singularities of solutions of (7) which behave locally like $(1+3w\,+4\,w^2) \cdot \ln(1+3w\,+4\,w^2) \cdot q_1 \,+\, q_2$, where q_1 and q_2 are analytic functions near the two quadratic roots $1+3w\,+4\,w^2 \;=\; 0$. This “weakly” singular behavior can also be seen on the monodromy matrix (see below) associated with these two roots, where one finds a nilpotent matrix of order two (no log^2 term).

Details on the local solutions of (7) around each regular singular point, together with the constants giving the particular physical solution $\tilde{\chi}^{(3)}$, will be given in a forthcoming publication.

We will sketch here the analysis of the Fuchsian equation by focusing on the local solutions of (7) around $w \;=\; 0$. Recall that the exponents near $w \;=\; 0$ are respectively: $9,\, 3,\; 2\,,1$ and again $2\,,1,\, 1$, all of them integers. Besides the $\tilde{\chi}^{(3)}/8$ solution of the Fuchsian differential equation which behaves, near $w \;=\; 0$, like (6), (which we will denote, from now on, S_9, since its leading term behaves like w^9), one expects solutions with leading terms behaving like w, w^2 and w^3, because of the previous integer exponents near $w \;=\; 0$. Furthermore, because of the singularity confluence (repetition of same exponents) and the occurrence of integer exponents, one also expects solutions with the above-mentioned logarithmic terms.

Actually, we have found two remarkable *rational and algebraic* solutions of (7), namely :

$$S_1 \;=\; \frac{w}{1\,-4\,w}, \qquad\qquad S_2 \;=\; \frac{w^2}{(1-4\,w)\,\sqrt{1-16\,w^2}} \tag{12}$$

a solution behaving like w^3, that we denote S_3 :

$$\begin{aligned} S_3 \;=\; & w^3 + 3\,w^4 + 22\,w^5 + 74\,w^6 + 417\,w^7 + 1465\,w^8 \\ & + 7479\,w^9 + 26839\,w^{10} \,+ \cdots \end{aligned} \tag{13}$$

and three solutions $S_1^{(2)}$, $S_2^{(2)}$ and $S_1^{(3)}$ with logarithmic terms, behaving, at small w, as:

$$S_1^{(2)} \;=\; w \cdot \ln(w) \cdot c_1 \,-32\,w^4 \cdot c_2 \tag{14}$$

$$S_2^{(2)} \;=\; w^2\,\ln(w) \cdot d_1 \,+\, 8\,w^4 \cdot d_2 \tag{15}$$

$$S_1^{(3)} \;=\; 3\,w \cdot \ln(w)^2 \cdot e_1 \,-12\,w^3 \cdot \ln(w) \cdot e_2 \,-19\,w^4 \cdot e_3 \tag{16}$$

where $c_1, c_2, d_1, d_2, e_1, e_2, e_3$ denote functions that are analytical at $w = 0$. However, these functions are not all independent. They can also be written in terms of S_1, S_2, S_3 and S_9 as follows :

$$\begin{aligned} S_1^{(2)} &= \ln(w) \cdot (S_1 - 4\,S_2 + 16\,S_3 - 216\,S_9) - 32\,w^4 \cdot c_2 \\ S_2^{(2)} &= \ln(w) \cdot (S_2 - 2\,S_3 + 24\,S_9) + 8\,w^4 \cdot d_2 \\ S_1^{(3)} &= 3\,\ln(w)^2 \cdot (S_1 + 5\,S_2 - 2\,S_3) \\ &\qquad - 6\,\ln(w) \cdot (2\,S_3 - S_1^{(2)} - 9\,S_2^{(2)}) - 19\,w^4 \cdot e_3 \end{aligned} \tag{17}$$

where :

$$\begin{aligned} c_2 &= 1 + \frac{167}{96}\,w + \frac{2273}{96}\,w^2 + \frac{6977}{120}\,w^3 + \frac{19371}{40}\,w^4 + \cdots \\ d_2 &= 1 + \frac{5}{2}\,w + \frac{103}{4}\,w^2 + \frac{315}{4}\,w^3 + \frac{2191}{4}\,w^4 + \cdots \\ e_3 &= 1 + \frac{7693}{456}\,w + \frac{575593}{11400}\,w^2 + \frac{2561473}{5700}\,w^3 + \frac{127434803}{93100}\,w^4 + \cdots \end{aligned}$$

Denoting $\Omega = 2\,i\,\pi$, one immediately deduces from the previous relations (17), the monodromy around $w = 0$:

$$\begin{aligned} S_1^{(2)} &\rightarrow S_1^{(2)} + \Omega \cdot (S_1 - 4\,S_2 + 16\,S_3 - 216\,S_9) \\ S_2^{(2)} &\rightarrow S_2^{(2)} + \Omega \cdot (S_2 - 2\,S_3 + 24\,S_9) \\ S_1^{(3)} &\rightarrow S_1^{(3)} - 6\,\Omega \cdot (2\,S_3 - S_1^{(2)} - 9\,S_2^{(2)}) \\ &\qquad + 3\,\Omega^2 \cdot (S_1 + 5\,S_2 - 2S_3) \end{aligned}$$

This calculation is a straight consequence of the fact that one only has a "logarithmic" monodromy, which just amounts to changing $\ln(w)$ into $\ln(w) + \Omega$, in the previous expressions. Denoting Id_7 the 7×7 identity matrix, and taking the following order $S_1,\, S_2,\, S_3,\, S_9,\, S_1^{(2)},\, S_2^{(2)},\, S_1^{(3)}$ for our seven-dimensional basis of solutions of (7), one can see that any n-th power of the 7×7 monodromy matrix M, $M_n = M^n$, satisfies $M_n^3 - 3\,M_n^2 + 3\,M_n - Id_7 = 0$, and can be written as the following sum of the identity matrix, and the two first powers of the order-three nilpotent matrix N ($N^3 = 0$) :

$$M^n = Id_7 + n \cdot \Omega \cdot N + \frac{(n \cdot \Omega \cdot N)^2}{2} = \exp(n \cdot \Omega \cdot N), \tag{18}$$

$$\text{where :} \qquad N = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 16 & -2 & -12 \\ 0 & 0 & 0 & 0 & -216 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 54 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As it should, one can check that the form (18), valid for any positive *or negative* integer n, is such that : $M^n \cdot M^m = M^{m+n}$.

The monodromy matrices corresponding to the other regular singular points yield very similar results. Those corresponding to $w = \pm 1/4$ are matrices of determinant -1, while all the others are of determinant $+1$. The monodromy around the 28 roots of P_7 is of course trivial (apparent singularities). The monodromy matrices corresponding to the other regular singular points verify very simple relations like the previous relation $M_n^3 - 3\,M_n^2 + 3\,M_n - Id_7 = 0$ or $M_n^2 - 2\,M_n + Id_7 = 0$, and also $M_n^6 - 3\,M_n^4 + 3\,M_n^2 - Id_7 = 0$ for those corresponding to $w = \pm 1/4$. This is a straight consequence of the fact that, in a Jordan form, only very simple Jordan blocks occur†, such as :

$$J_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad J_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

For instance, the monodromy matrices around the points $w = 0$ and $w = \infty$ are on the same footing, both requiring one J_3 Jordan block, one J_2 block, and the 2×2 identity matrix. There is another set of monodromy matrices corresponding to the points $w = 1$, $w = -1/2$ and the two roots of $1 + 3\,w + 4\,w^2 = 0$ (associated with J_2 blocks). More precisely, the monodromy matrices for $w = 0$ and $w = \infty$ verify $M_n^3 - 3\,M_n^2 + 3\,M_n - Id_7 = 0$, but for $w = -1/2$ and $w = 1$ they verify $M_n^2 - 2\,M_n + Id_7 = 0$, as well as for the two roots of $1 + 3\,w + 4\,w^2 = 0$.

A detailed analysis of the monodromy group will be performed elsewhere. However, such an analysis of the Galois group‡ becomes easier to perform when taking into account some remarkable factorization and decomposition properties that are sketched in the next section.

4. Algebraic properties of the Fuchsian equation

Let us define :

$$\lambda = $$
$$w^2\,\left(1 + 3\,w + 4\,w^2\right)^5\,(1 + 2\,w)^3\,(-1 + 4\,w)^{47/2}\,(1 + 4\,w)^{31/2}\,(1 - w)^3$$
$$f = (1 - w)\,(1 + 2w)\,(1 - 4w)^5\,(1 + 4w)^3\,(1 + 3w + 4w^2)$$

We denote L_7 the seventh order linear differential operator, corresponding to the Fuchsian differential equation (7):

$$L_7 = \frac{d^7}{dw^7} + \frac{1}{a_7(L_7)} \cdot \sum_{k=0}^{6} a_k(L_7)\,\frac{d^k}{dw^k} \tag{20}$$

where the $a_k(L_7)$'s are the polynomials a_k defined in (8). Polynomial $a_7(L_7)$ reads $a_7(L_7) = w^7 \cdot f \cdot P_7$, where we distinguish the "actual" and the apparent singularities.

We now give some remarkable factorization properties of the linear differential operator L_7. The fact that the very simple expression S_1 (see (12)) is a solution of the differential operator L_7 and that it is also solution of the first order differential operator L_1 :

$$L_1 = \frac{d}{dw} - \frac{1}{w\,(1 - 4\,w)} \tag{21}$$

† We thank Jacques-Arthur Weil for this result.

‡ The main difficulty is to find a *global* structure like the monodromy Galois group from the knowledge of all these *local* monodromy matrices expressed in the different well-suited *local* basis associated with each regular singular point.

implies the following factorization of L_7 (or more precisely the right-division of L_7 by L_1):

$$L_7 \; = \; M_6 \cdot L_1 \tag{22}$$

Similarly, the fact that S_2 is a solution of L_7, and that it is also clearly solution of a first order differential operator N_1 :

$$N_1 \; = \; {{d} \over {dw}} \; - \; {{2\,(1+2\,w)} \over {w\,(1-16\,w^2)}} \tag{23}$$

implies the existence of another factorization of L_7 :

$$L_7 \; = \; N_6 \cdot N_1 \tag{24}$$

The linear differential operators of order 6 read (X_6 denoting M_6 or N_6) :

$$X_6 \; = \; {{d^6} \over {dw^6}} \; + \; {{1} \over {a_6(X_6)}} \cdot \sum_{k=0}^{5} a_k(X_6)\, {{d^k} \over {dw^k}} \tag{25}$$

where $a_k(X_6)$ are polynomials in w and $a_6(M_6) \; = \; a_6(N_6) \; = \; w^4 \cdot f \cdot P_7(w)$.

These two factorizations are consequences of the existence of remarkable simple algebraic solutions of L_7. Besides this, there exists another factorization of L_7 related to the adjoint differential equation of (3.1). This is explained as follows. One can also see that the adjoint of L_7, denoted L_7^*, admits the following rational solution§:

$$S_1^* \; = \; {{f \cdot Q_6(w)} \over {w^3 \cdot P_7}} \tag{26}$$

with :

$$\begin{aligned} Q_6 \; = \; & 1+19\,w-368\,w^2-3296\,w^3+17882\,w^4+272599\,w^5 \\ & +160900\,w^6-6979208\,w^7+7550800\,w^8+203094872\,w^9 \\ & -278920192\,w^{10}-3959814304\,w^{11}-2115447424\,w^{12} \\ & +20894729472\,w^{13}+39719728128\,w^{14}+20516098048\,w^{15} \\ & +256763363328\,w^{16}-327065010176\,w^{17}-8810227761152\,w^{18} \\ & +414933057536\,w^{19}+116411936538624\,w^{20} \\ & +296827723186176\,w^{21}+317648030138368\,w^{22} \\ & +179148186189824\,w^{23}+194933533179904\,w^{24} \\ & +112931870081024\,w^{25}-55246164328448\,w^{26} \\ & +11063835754496\,w^{27}+1511828488192\,w^{28} \end{aligned} \tag{27}$$

yielding immediately the following factorization (or more precisely the left-division of L_7 by M_1):

$$L_7 \;\; = \;\; M_1 \cdot L_6, \qquad\qquad \hbox{where :} \qquad L_6 \; = \; L_5 \cdot N_1 \tag{28}$$

$$\hbox{with:} \qquad\qquad M_1 \; = \; {{d} \over {dw}} \; + \; {{1} \over {S_1^*}}\, {{dS_1^*} \over {dw}} \tag{29}$$

and where L_5 and L_6 are linear differential operators of order five and six respectively, which read :

$$L_q \; = \; {{d^q} \over {dw^q}} \; + \; {{1} \over {a_q(L_q)}} \cdot \sum_{k=0}^{q-1} a_k(L_q)\, {{d^k} \over {dw^k}}, \qquad\qquad q = 5, 6 \tag{30}$$

§ We thank Jacques-Arthur Weil for the remarkable result (26) and (28).

with :

$$a_6(L_6) \; = \; w^6 \cdot f \cdot Q_6(w)$$
$$a_5(L_6) \; = \; w^5 \cdot (1-4\,w)^4\,(1+4\,w)^2 \cdot Q_5(w) \qquad (31)$$
$$a_5(L_5) \; = \; w^3 \cdot (1-4\,w)\,(1+4\,w)^2 \cdot f \cdot Q_6(w)$$

The roots of polynomial $Q_6(w)$ are apparent singularities of the differential equations associated to the operators L_6 and L_5.

One can also see that M_6 in (22) can also be decomposed as follows :

$$M_6 \; = \; M_5 \cdot T_1 \qquad \text{with:}$$
$$M_5 \; = \; \frac{d^5}{dw^5} \, + \, \frac{1}{a_5(M_5)} \cdot \sum_{k=0}^{4} a_k(M_5)\,\frac{d^k}{dw^k}$$

where $\; a_5(M_5) \; = \; w^4\,(1-4\,w)^5\,(1+4\,w)^4 \cdot f \cdot P_7(w), \quad$ and:

$$T_1 \; = \; \frac{d}{dw} \, - \, \frac{1+4\,w+48\,w^2}{w\,(1-16\,w^2)}$$

Calculations seem to show that L_5 (resp. M_5) is irreducible : we have not found any further factorizations like $L_5 \, = \, L_4 \cdot A_1$ or $L_5 \, = \, B_1 \cdot L_4$, or even $L_5 \, = \, L_3 \cdot A_2$ or $L_5 \, = \, B_2 \cdot L_3$.

One can also be interested in the Wronskians of L_7 and L_6. They can be written as follows :

$$W(L_7) \; = \; \frac{P_7}{\lambda}, \qquad\qquad W(L_6) \; = \; \frac{f \cdot Q_6(w)}{w^3 \cdot \lambda} \qquad (32)$$

thus checking the following relation on the Wronskians of L_7, L_6, and M_1, deduced from (28), namely:

$$W(L_7) \; = \; W(M_1) \cdot W(L_6) \qquad (33)$$

From these factorizations, and/or decompositions, relations, corresponding to the existence of *rational, or algebraic* (square root of rational), expressions, one should not be surprised to find that the Wronskians of all the various operators we define here, are remarkable rational, or algebraic (square root of rational), expressions (see (32)). The fact that the squares of the Wronskians (32) are rational functions corresponds to the following identities between P_7 and P_6 (resp. Q_6 and Q_5) :

$$\frac{P_7'}{P_7} \, - \, \frac{\lambda'}{\lambda} + \frac{(1-4\,w)^4\,(1+4\,w)^2}{w\,f} \cdot \frac{P_6}{P_7} \; = \; 0 \qquad (34)$$

$$\frac{Q_6'}{Q_6} \, - \, \frac{3}{w} \, + \, \frac{f'}{f} - \frac{\lambda'}{\lambda} + \frac{(1-4\,w)^4\,(1+4\,w)^2}{w\,f} \cdot \frac{Q_5}{Q_6} \; = \; 0 \qquad (35)$$

The Wronskians (or their inverse for the adjoints) of all the differential operators L_7, L_7^*, M_6, M_6^*, M_5, M_5^* can be expressed as simple powers of P_7 divided by a simple expression similar to λ in (20). Similarly, the Wronskians (or their inverse for the adjoints) of all the differential operators L_6, L_6^*, L_5, L_5^* can be expressed as simple powers of Q_6 divided by a simple λ-like expression. All these Wronskians are thus *rational* (L_5, L_5^*, M_5, M_5^*) or such that their *square is rational* (L_7, L_7^*, L_6, L_6^*, M_6, M_6^*). Of course, there are relations between these Wronskians which are in agreement with the operator factorizations previously described (see (33)).

Besides the "fundamental" singularities $w \; = \; 0, -1/2, \pm 1/4, 1$ and the "unexpected" quadratic numbers singularities solutions of $1+3\,w+4\,w^2 \; = \; 0$, one

could say that a large part of the "arithmetic complexity" of the operator L_7 arises from the two polynomials of degree 28 in w, P_7 and Q_6.

The occurrence of the *apparent singularities*, associated with the quite large polynomial P_7, can be considered slightly unpleasant or disturbing : one would like to exchange the seventh order Fuchsian equation (7), for another differential equation (or a differential system) where these "spurious" singularities have disappeared. This is the so-called *desingularization problem* for differential equations [17, 18]. We have performed such "desingularization" : the "price" to be paid is that one has no longer a seventh order differential equation, but an *eighth* order differential equation. The associated eighth order differential operator reads:

$$L_8 \; = \; {{d^8} \over {dw^8}} \; + \; {{1} \over {a_8(L_8)}} \cdot \sum_{k=0}^{7} a_k(L_8) \, {{d^k} \over {dw^k}} \qquad \hbox{with :} \qquad (36)$$

$$a_8(L_8) \; =$$

$$w^{n_1} \; (1-4\,w)^{n_2} \; (1+4\,w)^{n_3} \; (1+2\,w)^{n_4} \; (1-w)^{n_5} \; \left(1+3\,w+4\,w^2\right)^{n_6}$$

$n_1, \cdots, n_6$ being positive integers. The other polynomials $a_k(L_8)$, are quite large and will not be given here. Let us sketch this desingularization procedure. Recall the exponents associated with the roots of P_7 (see Table 1). They are $0, 1, 2, 3, 4, 5, 7$; one sees that the exponent 6 is missing. Basically, the method amounts to building a differential equation having all the solutions of (7), together with a solution†† associated with this missing exponent 6, in order to "fill the gap" and make the roots of P_7 ordinary points for the homogenous differential equation associated with L_8. Note too that the desingularized differential equation, or equivalently the operator L_8 are far from being unique. The various eighth order L_8 operators are, by construction, of the form $L_8 \; = \; \tilde{L}_1 \cdot L_7$, where the first order operator $\tilde{L}_1$ is quite involved¶. From a desingularized form like (36) one can also introduce a differential system$^+$:

$$\theta \cdot Y \; = \; A \cdot Y, \qquad\qquad Y \; = \; (y, \; \theta \cdot y, \; \theta^2 \cdot y, \; \theta^3 \cdot y, \; \cdots) \qquad (37)$$

$$\hbox{with :} \qquad A \; = \; {{A_0} \over {w}} \; + \; {{A_1} \over {w-1/4}} \; + \; {{A_2} \over {w+1/4}} \; + \; {{A_3} \over {w+1/2}}$$

$$\qquad + \; {{A_4} \over {w-1}} \; + \; {{A_5\,w+A_6} \over {w^2+3/4\,w+1/4}} \; + \; P(w) \cdot Id_8$$

where Id_8 denotes the 8×8 identity matrix, $P(w)$ denotes a polynomial in w, and where the A_n matrices are very simple 8×8 matrices, simply related to the monodromy matrices, and where θ denotes the "well-suited" derivation associated with the "true" singularities :

$$\theta \; = \; w \cdot (1-w) \; (1+2\,w) \; (1-4\,w) \; (1+4\,w) \; \left(1+3\,w+4\,w^2\right) \cdot {{d} \over {dw}}$$

A form like (37) is clearly much simpler, and more canonical : the calculations one has in mind (Galois group, rigidity index (see below)) should be much simpler to perform with this canonical system form (37). Details will be given elsewhere.

††The solution $S \; = \; P_7^6$ is too naive, it puicks out the apparent singularities of P_7 but introduces new spurious apparent singularities.

¶ As far as the solutions of L_8 are concerned, the differential operator L_8 adds to the known solutions of L_7, a solution with a strong exponential behavior.

$^+$ See for instance the chapter 6 of [19].

5. Comments and Speculations

A better understanding of the total susceptibility χ certainly requires an exhaustive knowledge of the singularities of the successive Fuchsian differential equations associated with the n-particle contributions $\chi^{(n)}$. Besides the apparent singularities associated with the roots of the polynomial P_7, we have noted the occurrence of the two rather unexpected *quadratic numbers* solutions of $1+3\,w+4\,w^2 \,=\, 0$, which, also, *are not of Nickel's form* (11). The elliptic parameterization of the Onsager model is well-known. Recalling the exact expression of the modular invariant $\mathcal{M}$ for the Ising model (see for instance [20]) :

$$\mathcal{M} \,=\, \frac{1}{1728}\,\frac{\left(1-16\,w^2+16\,w^4\right)^3}{w^8\,(1\,-16\,w^2)} \tag{38}$$

it can be seen that the two *quadratic numbers* correspond to a *rational value* of the modular invariant: $\mathcal{M} \,=\, -125/64$, while $w \,=\, 1$ also corresponds to a rational value $\mathcal{M} \,=\, -1/25920$, that $w \,=\, \pm 1/4,\, 0,\, \infty$ correspond to $\mathcal{M} \,=\, \infty$, and that $w \,=\, -1/2$ corresponds to the rational value $\mathcal{M} \,=\, 32/81$. Could this mean that the Fuchsian equation (7) could be "canonically" associated with an elliptic curve ? One can, for instance, recall the period mappings and Picard-Fuchs equations (and beyond mirror symmetries ...) associated with a family of elliptic curves [21, 22]. The Picard-Fuchs equation :

$$144\,s\,(s-1)^2\cdot\left(s\cdot\frac{d^2 f}{ds^2}\,+\,\frac{d\,f}{ds}\right)+\,(31\,s\,-4)\cdot f \,=\, 0 \tag{39}$$

is associated with the family of elliptic curves :

$$y^2 \,=\, 4\,x^3\,+\frac{27}{1-s}\cdot x\,+\frac{27\,s}{1-s}$$

We now make a few comments on the physical solution S_9. The Fuchsian equation is highly non-trivial and structured : from the previous factorizations, and/or decompositions, one might imagine that $S_9 \,=\, \tilde{\chi}^{(3)}/8$ is in fact a solution of a sixth order homogeneous differential equation, or even a fifth order homogeneous differential equation. This is not the case. Relation (28) means that $S_9 \,=\, \tilde{\chi}^{(3)}/8$ can be seen as a linear combination of S_1 and of a solution of a sixth order linear homogeneous differential equation, $S(L_6)$, associated with the operator L_6. The solution $S_9 \,=\, \tilde{\chi}^{(3)}/8$ is actually such a linear combination $S_9 \,=\, \alpha\cdot S_1\,+\,S(L_6)$, α being different from zero*. The three-particle contribution, $\tilde{\chi}^{(3)}$, is *thus a solution of the seventh order differential equation* (7) *and not of a homogeneous linear differential of smaller order.*

Coming back to the analysis of the Galois monodromy group [23, 24, 25] of the Fuchsian equation (7), it is clear that the previous factorizations, and/or decompositions, impose severe reductions of the group. From these various factorizations, or decompositions, of the linear operator $L_7 \,=\, M_1\cdot L_6 \,=\, M_1\cdot L_5\cdot N_1$, one can deduce that the Galois monodromy group is isomorphic to $Gal(L_6)$, the Galois monodromy group of L_6. Introducing $Gal(L_5)$, the Galois monodromy group of L_5, one deduces from $L_6 \,=\, L_5\cdot N_1$, that the operator N_1 injects $Gal(L_5)$ in $Gal(L_6)$. This does not mean that $Gal(L_6)$ is isomorphic to $Gal(L_5)$, but that knowledge of $Gal(L_5)$ is required to describe $Gal(L_6)$. Recalling the rationality of the Wronskian

* The coefficient α characterizing the "projection" of S_9 on S_1, can easily be calculated writing $L_6(S_9\,-\alpha\cdot S_1) \,=\, 0$. One finds $\alpha = 1/24$.

of L_5, one deduces that $Gal(L_5)$ is a subgroup of $SL(5,C)$ and not $GL(5,C)$ (the rationality of the Wronskian means that all the monodromy matrices of L_5 have $+1$ determinants). The monodromy matrices of L_6 (as well as the one of the Fuchsian equation (7)) have ± 1 determinants. The Galois group $Gal(L_6)$ is, up to a Z_2-graduation, a subgroup of $SL(6,C)$. Therefore the Galois group of the Fuchsian equation (7) is represented by 6×6 matrices with ± 1 determinants, most of its structure requiring the analysis of a $Gal(L_5)$ subgroup of $SL(5,C)$. Of course, one can also understand the Galois group of the Fuchsian equation (7) from the analysis of the Galois group of M_5, namely $Gal(M_5)$ which is also subgroup of $SL(5,C)$ (isomorphic to $Gal(L_5)$). It seems that L_5 is not self-adjoint♯ (modulo conjugation by an operator) and irreducible (see Section 4). The two facts seem to rule out the existence of a symplectic structure. A more detailed analysis of the Galois group of the Fuchsian equation (7), that is, of the Galois groups $Gal(L_5)$, and $Gal(L_6)$, will be given in forthcoming publications.

Let us, now, focus again on the "physical" solution of the Fuchsian equation (7), $S_9 = \tilde{\chi}^{(3)}/8$, and on its successive derivatives S_9', S_9'', $S_9^{(3)}, \cdots$ We have seen that, as far as the *linear* dependence of these expressions is concerned, we have a *seven-dimensional* vector space (S_9 is *not* a solution of a homogeneous sixth order differential equation†). However, to better understand the "true nature" of the susceptibility χ, one would like to characterize the "degree of transcendence" of $S_9 = \tilde{\chi}^{(3)}/8$, that is the minimal number of successive derivatives of S_9 satisfying an algebraic *non-linear* relation+. The Galois monodromy group gives a valuable information about this "degree of transcendence". Let us consider the orbit of S_9 under the Galois monodromy group: one gets an algebraic variety dense in the subspace of solutions. The dimension of this algebraic variety is actually this degree of transcendence. This analysis, however, requires an exact knowledge of the Galois monodromy group.

The susceptibility χ has been shown to be a transcendental (non-holonomic, non D-finite) function: it cannot be solution of a *linear* differential equation, but this does not mean that it cannot be solution of a differential equation. Along the previous "non-linear" line, one should emphasize that the possibility that χ could be solution of some (Painlevé-like ?) *non-linear* differential equation, is not yet totally ruled out !

Within the "linear" Picard-Vessiot Galois monodromy group framework, one can also try to evaluate the "index of rigidity [26, 27]" of our Fuchsian differential equation, or the "index of rigidity" of the operators L_6 or L_5. Roughly speaking, this index of rigidity corresponds to the number of parameters that can be introduced to deform the linear differential equation, keeping the local monodromy matrices fixed‡. For instance, the hypergeometric functions are totally rigid : as a consequence, "almost everything" can be calculated on such functions§. The differential operators L_7, L_6 or L_5 have many integer exponents, and, thus, their solutions have logarithmic behaviors $(w - w_0)^n \cdot \big(ln(w - w_0)\big)^m$ around each regular singular point w_0. However, recalling

♯ However it can be shown that L_6 is not self-adjoint (modulo conjugation by an operator). L_6 has S_2 as a solution. If it were self-adjoint, L_6^* would have a similar quadratic solution which is not the case.

† To be totally rigorous one should add that the minimal operator of S_9 is a factor of L_7, and since these factors are of order 1 or 6, and that S_9 does not vanish on these factors, it must vanish on L_7.

+ For instance the Weierstrass $\mathcal{P}$ function verifies the non-linear relation $\mathcal{P}'^2 = 4\,\mathcal{P}^3 + g_2\,\mathcal{P} + g_3$.

‡ See the notion of *rigid local systems* [27, 28]. Basically it amounts to calculating the dimensions of the centralizers of all the monodromy matrices.

§ In the case of totally rigid systems, N. Katz has shown that the solutions have a *geometric interpretation* : they can be seen as periods of some algebraic varieties [27].

(14), (15) and (16), and, more generally, considering the behavior near all the singular points, one sees that *only* $ln(w - w_0)$ and $\big(ln(w - w_0)\big)^2$ behaviors occur*, thus *downsizing* (see for instance (4.31) in Chapter (4) of [19]) the *index of rigidity* of these differential equations. This is a strong indication that the Fuchsian differential equation (7), or the differential equations associated with L_6 or L_5, *are extremely rigid*. This can also be seen on the various monodromy matrices. The calculation of the index of rigidity, or in other words, the calculation of the "small" number of deformation parameters of (7) (resp. L_6, L_5) will be performed elsewhere. This remarkable rigidity is not a surprise, when recalling the well-known isomonodromy theory of the Ising model, and in particular the occurrence of Painlevé equations for the correlation functions of the Ising model. Does the deformation theory of (7) in this "small" number of deformation parameters actually yield Painlevé-like equations ? This is an open question. From a more down-to-earth viewpoint, this rigidity can be seen as "inherited" from the "total" rigidity of hypergeometric functions: our calculations for generating the w-series of $\tilde{\chi}^{(3)}$ are actually "flooded" by linear combinations (with binomial coefficients) of products of hypergeometric functions (see [15] for more details).

6. Conclusion

The linear differential equation we have found for the three-particle contribution, $\chi^{(3)}(w)$, to the susceptibility of the square lattice Ising model, is a highly structured and remarkable Fuschsian differential equation. We have sketched many of its remarkable properties and symmetries. It is also worth recalling that the 28 roots of the polynomial P_7 *are apparent singularities*. We have been able to desingularise the Fuchsian equation (7), in order to get rid of these apparent singularities related to the quite involved polynomial P_7. A deeper analysis of this Fuchsian equation (monodromy group, critical behavior of the solutions around the various singular points, ...) will be given in forthcoming publications.

All these results can be generalized, *mutatis mutandis*, to deduce the Fuchsian equations corresponding to the other n-particle contributions $\chi^{(n)}$'s. The building of a computer program with a polynomial growth algorithm which can be generalized, mutatis mutandis, for the other $\chi^{(n)}$'s, was the key ingredient to get our Fuchsian equation. The ideas developed to create such polynomial growth programs underline the role played by hypergeometric functions, coming from a large number of remarkable identities on the underlying variables of the problem. One may also think that quite complicated "fusion-type" relations on these hypergeometric functions can exist. This is crystal clear in the case of the 2-particle contribution $\chi^{(2)}$ [15]. This however remains to be done in full.

Clearly, beyond $\chi^{(3)}$, a global understanding of the structure of the hierarchy of all the $\chi^{(n)}$'s could be contemplated. A better understanding of the total susceptibility χ certainly requires an exhaustive knowledge of the singularities of the successive n-particle contributions $\chi^{(n)}$, or equivalently, of the corresponding successive Fuchsian equations. As far as analytical properties are concerned, we saw the occurrence in (7) of unexpected quadratic numbers singularities, $1 + 3w + 4w^2 = 0$ (these two singularities are not on the s-unit circle: $|s| = \sqrt{2}$, $1/\sqrt{2}$). Curiously these unexpected singularities correspond to a *rational* value of the *modular invariant*. Could this mean

* This can be seen directly on the very simple Jordan blocks for the monodromy matrices (see (19)).

that the Fuchsian equations for the successive $\chi^{(n)}$'s are "canonically¶" associated with an elliptic curve ? Should we rather understand this hierarchy of Fuchsian equations in a linear (or non-linear) *monodromy deformation theory* framework ?

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¶ See for instance the Picard-Fuchs equation (39) associated with a family of elliptic curves [21, 22].